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Note

On a Vortex Sheet Approach to the Numerical Calculation of Water Waves

I. Introduction

The past few years have seen considerable progress in the development of sophisticated numerical methods for the calculation of plane waves in shallow water. Of these, the Stanford University Modified Marker and Cell (SUMMAC) Eulerian finite difference method [1] is probably the most highly developed. The LINC method [2], based upon a Lagrangian formulation, also appears to be quite powerful and promising.

At the same time there has been continued interest in methods involving expansions in powers of the vertical distance from the bottom. This procedure permits the elimination of the vertical space variable, and hence a reduction to only one space variable plus time. It leads to simple wave equations, Airy's long wave theory, or the Korteweg–de Vries theory, depending upon the size of the Ursell parameter; these ideas are reviewed and extended by Madsen and Mei [3]. Other authors (e.g. [4, 5]), proceeding more formally, have chosen to build upon Airy's model by including the vertical acceleration in various approximate ways.

The object of the present note is to formulate a rather different line of approach, one based upon a vortex-sheet representation of the free surface and the bottom boundary [6], and applicable to arbitrarily large, though nonbreaking, inviscid, irrotational, incompressible, two-dimensional water waves. The principal advantage of this "surface" formulation is, again, the accompanying reduction from two space dimensions to one, but this time Laplace's equation on the velocity potential is satisfied automatically and exactly. In return for this favor the resulting equations are of an integrodifferential character, but this complication can be suppressed and the equations integrated in finite time steps.

II. Vortex Sheet Formulation

From classical potential theory, the flow within the water can be generated by a distribution of circulation on the free surface $\mathscr{S}$, of lineal density $G^w(x, t)$ per unit x-length say, together with a distribution $\bar{G}(x, t)$ on the bottom, or beneath the bottom if an image system is available, where the x axis coincides with the undisturbed free surface and y is measured upwards. Similarly, the flow of air above the water, assumed irrotational, can be generated by a distribution of vorticity over $\mathscr{S}$. It is more convenient, however, to generate the airflow by a distribution $G^a(x, t)$, say, over $\mathscr{S}$, *together with* the distribution $\bar{G}(x, t)$ over the bottom, since then the required continuity of the normal velocity across the free surface implies that $G^a(x, t) = G^w(x, t) \equiv G(x, t)$, say, so that the *single* vortex system, consisting of the combination of G and $\bar{G}$, accounts for the flow in both the air and water. (Note that G here corresponds to $2\pi G$ of [6]).

Now, with Laplace's equation on the velocity potential ϕ, continuity of the normal velocity across $\mathscr{S}$, and the bottom boundary condition all satisfied, it remains to compute the vortex density $G(x, t)$ and the free surface height $Y(x, t)$ in accordance with the kinematic and dynamic free surface conditions on $\mathscr{S}$,

$$Y_t = v^a - u^a Y_x \qquad \text{or,} \qquad v^w - u^w Y_x \tag{1}$$

and

$$p^a = p^w \equiv p, \tag{2}$$

respectively, where subscripts denote partial differentiation, the a, w superscripts refer to the air and water, u and v are the x, y velocity components, and the effects of surface tension are omitted. Averaging Eqs. (1), we have instead

$$Y_t = V - UY_x, \tag{3}$$

where $U(x, t) \equiv (u^a + u^w)/2$ on $y = Y$, and similarly for $V(x, t)$; U, V may be regarded as the components of the "velocity of the free surface vortices." Besides the U, V decomposition, we will also use the tangential and normal components T and N (see Fig. 1).

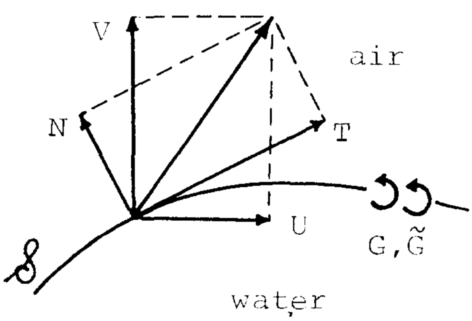

Fig. 1. Velocity of the free surface vortices.

To express (2) in terms of our principal variables G, Y, U, V we start with the Bernoulli equation for the air and water sides of $\mathscr{S}$,

$$\phi_3^{a,w}(x, Y, t) + \tfrac{1}{2}[(T \mp \tilde{G}/2)^2 + N^2] + gY + p^{a,w}(x, Y, t\,/\rho^{a,w} = \mathscr{F}^{a,w}(t), \quad (4)$$

where $\tilde{G}$ is the circulation per unit arc length (so that $\tilde{G}\,ds = G\,dx$), g is the acceleration of gravity, $\rho^{a,w}$ are the mass densities of the air and water, $\mathscr{F}^{a,w}$ are the "Bernoulli constants" for the air and water, and where we have noted that the tangential surface velocities of the fluid particles are $T \mp \tilde{G}/2$ with the upper and lower signs corresponding to the air and water, respectively. The subscripted 3 denotes differentiation with respect to the third argument; this notation will be used whenever a letter subscript is rendered ambiguous by the x and t dependence in the Y argument.

Now, defining $\phi^{a,w}[x, Y(x, t), t] \equiv \Phi^{a,w}(x, t)$, we have

$$\Phi_x^{a,w} = \phi_1^{a,w} + \phi_2^{a,w} Y_x = U \mp \tilde{G}C/2 + (V \mp \tilde{G}S/2)\, Y_x = (T \mp \tilde{G}/2)/C \quad (5)$$

since $Y_x = S/C$ and $T = UC + VS$, where C, S are short for $\cos(\tan^{-1} Y_x)$, $\sin(\tan^{-1} Y_x)$, respectively. Similarly,

$$\Phi_t^{a,w} = \phi_3^{a,w} + \phi_2^{a,w} Y_t \quad (6)$$

so that

$$\phi_3^{a,w} = \Phi_t^{a,w} - (V \mp \tilde{G}S/2)\, Y_t\,. \quad (7)$$

Inserting this into (4), and differencing the equations for water and air, we have

$$\Phi_t^{w} - \Phi_t^{a} - \tilde{G}SY_t + T\tilde{G} + ((1/\rho^w) - (1/\rho^a))P = \mathscr{F}^w - \mathscr{F}^a, \quad (8)$$

where we have defined $p^{a,w}[x, Y(x, t), t] = p[x, Y(x, t), t] \equiv P(x, t)$. Finally, if we take $\partial/\partial x$ of (8) and note that $\Phi_{tx}^{a,w} = \Phi_{xt}^{a,w}$ where $\Phi_x^{a,w}$ is given by (5), then with the help of (3) and the relations $\tilde{G} = GC$ and $T = UC + VS$, we obtain the result

$$G_t + (UG)_x = ((1/\rho^a) - (1/\rho^w))\, P_x \quad (9)$$

To interpret (9) physically, consider the circulation

$$\Gamma_{AB}(t) = \int_{x_A(t)}^{x_B(t)} G(x, t)\, dx \quad (10)$$

between any two elementary vortices labeled A and B on $\mathscr{S}$. Leibnitz differentiation gives $d\Gamma_{AB}/dt \sim \Delta x[G_t + (UG)_x]$ as $x_B - x_A \equiv \Delta x$ tends to zero. Comparing this with (9), we see that Γ_{AB} will be conserved if the fluid densities are equal. Otherwise, the term on the right-hand side of (9) can not be expected to be zero, and there will be a generation or demise of circulation due to the tangential pressure gradient imparting different accelerations to the adjacent fluids by virtue of their disparate densities.

If, as before, we insert (7) into (4), but this time *add* the equations for air and water, and then take $\partial/\partial x$ of the result, we obtain

$$-2(U + VY_x)_t + (2VY_t - U^2 - V^2 - C^2G^2/4 - 2gY)_x = ((1/\rho^a) + (1/\rho^w))\, P_x\,. \tag{11}$$

Finally, eliminating P_x between (9) and (11), and noting that $C_x = -SC^2Y_{xx}$, leads to

$$\begin{aligned} G_t = &-(UG)_x + (\kappa/2)[C^2G(CSGY_{xx} - G_x) \\ &-4(U_t + UU_x) - 4(V_t + UV_x)\, Y_x - 4gY_x], \end{aligned} \tag{12}$$

where $\kappa = (\rho^w - \rho^a)/(\rho^w + \rho^a)$ and U, V can be given by Biot–Savart integrals over the vortex sheet $\mathscr{S}$ and its image $\bar{\mathscr{S}}$, say, beneath the bottom. Specifically,

$$U(x, t) = \frac{1}{2\pi} \int \left\{ \frac{Y(\xi, t) - Y(x, t)}{(\xi - x)^2 + [Y(\xi, t) - Y(x, t)]^2} + \frac{Y(x, t) - \bar{Y}(\xi, t)}{[\bar{\xi}(\xi, t) - x]^2 + [\bar{Y}(\xi, t) - Y(x, t)]^2} \right\} G(\xi, t)\, d\xi$$

and (13)

$$V(x, t) = \frac{1}{2\pi} \int \left\{ \frac{x - \xi}{(\xi - x)^2 + [Y(\xi, t) - Y(x, t)]^2} + \frac{\bar{\xi}(\xi, t) - x}{[\bar{\xi}(\xi, t) - x]^2 + [\bar{Y}(\xi, t) - Y(x, t)]^2} \right\} G(\xi, t)\, d\xi,$$

where $\bar{\xi}(\xi, t)$, $\bar{Y}(\xi, t)$ are the coordinates of the image point corresponding to the free surface point ξ, $Y(\xi, t)$.

Our "working equations," then, are (3) and (12), together with initial conditions $Y(x, 0)$ and $G(x, 0)$, and the Biot–Savart relations (13).

Before discussing the solution of this system of equations in the variables G, Y, U, V, we might note the difference between our dynamic boundary condition (2) and the customary condition $p = 0$, which completely ignores the air density and is equivalent to setting $\kappa = 1$ in (12). Of course, $p = 0$ is generally an excellent approximation for the air/water case; it's just that the vortex model automatically involves us with the airflow, so that (2) is more natural here.

III. Method of Solution

Now, the governing Eqs. (3), (12), and (13), are coupled, nonlinear, integrodifferential equations. Nevertheless, suppose we know $G(x, t)$ and $Y(x, t)$ at some instant t. Then we can compute $U(x, t)$ and $V(x, t)$ from (13) and, with the help of numerical differentiation, we can also compute Y_x, Y_{xx}, U_x, V_x, C, and S. Thus the right-hand sides of (3) and (12) can be determined, and hence $G(x, t + \Delta t)$ and $Y(x, t + \Delta t)$ are obtained by numerical integration—except for the fact that the $U_t(x, t)$ and $V_t(x, t)$ terms needed in (12) are *not* computable from the initial data $G(x, t)$ and $Y(x, t)$. In order to obtain values for U_t and V_t, we first iterate over a very small "virtual" time step δt, which is an order of magnitude smaller than the "nominal" step size Δt, as follows: Estimating U_t, V_t in (12), we integrate (3) and (12) (at a discrete set of x's) from t to $t + \delta t$ using Euler's method. With $Y(x, t + \delta t)$ and $G(x, t + \delta t)$ thus determined, $U(x, t + \delta t)$ and $V(x, t + \delta t)$ are computed from (13); comparing these with $U(x, t)$ and $V(x, t)$, we are thus able to form improved estimates of $U_t(x, t)$ and $V_t(x, t)$. Repeating our integration of (12) using these new estimates, we obtain improved values for $G(x, t + \delta t)$. The process is repeated until suitable convergence is attained; i.e., until $U(x, t) + U_t(x, t)\,\delta t$ and $V(x, t) + V_t(x, t)\,\delta t$ are sufficiently close to the Biot–Savart computed quantities $U(x, t + \delta t)$ and $V(x, t + \delta t)$, respectively.

The relatively large Δt step is accomplished by the more sophisticated Runge–Kutta–Merson scheme [6], where the "inner iteration" for U_t, V_t must be carried out five times for each Δt step.

IV. Initial Numerical Test; Solitary Wave

Although we are not yet in a position to make a definitive statement regarding stability, choice of step size, and so on, we *are* able to report the results of our initial computer runs, for the case of a solitary wave of "intensity" $\gamma =$ wave height/undisturbed depth $= 0.5$.

To obtain the starting conditions $Y(x, 0)$ and $G(x, 0)$ we first developed a five-term McCowan-type expansion [7] for the water flow; this provided us with Y and the tangential surface velocity T^w. However, in computing G as $(T^w - T^a)\, ds/dx$ we still had to compute the airflow over the wave in order to find T^a. This was done by means of a suitable distribution of doublets along the x axis; details are given in [6]. The resulting Y and G are shown in Fig. 2. Note that the tails of G decay rather slowly, only as $O(x^{-2})$, due to the airflow. That is whereas T^w decays exponentially in x, T^a drops off only as $O(x^{-2})$.

Thanks to the flat bottom, a simple reflection provided an exact image system. With the undisturbed depth $h = 1$, we selected an x range between -7 and $+7$

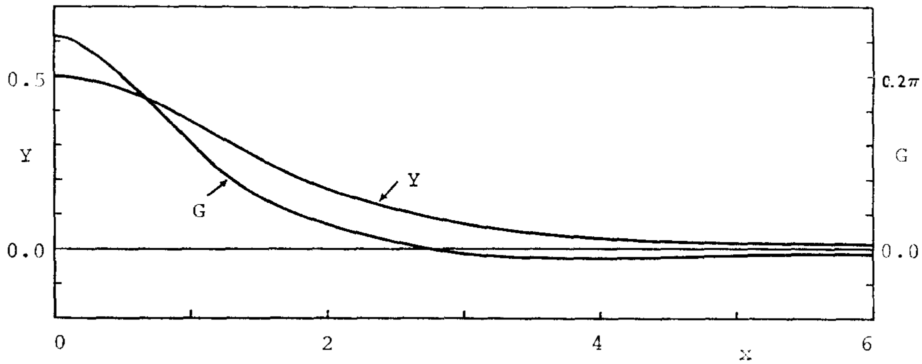

FIG. 2. Solitary wave height and vortex density for $\gamma = 0.5$.

with 59 x points, i.e., at intervals of 0.25. However, the Biot–Savart integrations were carried out from $-\infty$ to $+\infty$, with the tail portions ($-\infty$ to -7 and $+7$ to $+\infty$) done approximately and analytically by extrapolating G according to its known $O(x^{-2})$ behavior. The nominal time steps Δt were chosen to correspond to a wave movement of 0.25, and the crest was initially offset at $x = -1.5$. The program was run for 14 such steps (on the BRLESC machine at Aberdeen Research and Development Center, machine time being about 13 min), at which point the wave crest was at $x = 2$. The final wave shape is plotted in Fig. 2, and coincides with the initial shape; the discrepancy of 0.001 between initial and final wave heights, for example, is not observable in the plot. We terminated the calculation after only this modest amount of travel because the Biot–Savart tail integrations for $7 < x < \infty$ become increasingly suspect as the crest approaches $x = 7$.

We also can another case, a solitary wave running up on a beach, but hesitate to document those results here since various additional simplifications introduced for the sake of expediency, such as the resort to an approximate image system, render those results difficult to assess. In any case, they are available in [6].

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